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Light rays joining two submanifolds in space–times ★

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Abstract

Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a stationary Lorentz metric and P_0, P_1 be two closed submanifolds of $\mathcal{M}_0$. By using the Ljusternik–Schnirelman theory and variational tools, we prove the influence of the topology of P_0 and P_1 on the number of lightlike geodesics in $\mathcal{M}$ joining $P_0 \times \{0\}$ to $P_1 \times \mathbb{R}$.

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1. Definitions and statement of main results

Let $\mathcal{M}$ be a smooth finite dimensional manifold and $\langle \cdot, \cdot \rangle_z$ be a Lorentz metric on it, that is a smooth symmetric (0,2) tensor field on $\mathcal{M}$ which defines a non-degenerate bilinear form of index 1 on each tangent space $T_z\mathcal{M}$, $z \in \mathcal{M}$.

Let us recall that the geodesics in $\mathcal{M}$ are smooth curves $z : [a, b] \to \mathcal{M}$ such that

$$D_s \dot{z}(s) = 0 \quad \text{for all } s \in [a, b],$$

where D_s denotes the covariant derivative along z induced by the Levi–Civita connection of $\langle \cdot, \cdot \rangle_z$.

It is easy to prove that for each geodesic $z = z(s)$ the energy

$$E(z) = \langle \dot{z}(s), \dot{z}(s) \rangle_z$$

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is constant in $[a, b]$, so a geodesic $z = z(s)$ is timelike, lightlike or spacelike if $E(z)$ is negative, null or positive, respectively.

A Lorentz manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle_z)$ is called stationary if there exists a finite-dimensional Riemannian manifold $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_x)$ such that $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and $\langle \cdot, \cdot \rangle_z$ is given by

$$\langle \zeta, \zeta \rangle_z = \langle \xi, \xi \rangle_x + 2 \, \langle \delta(x), \xi \rangle_x \tau - \beta(x) \tau^2, \tag{1.1}$$

for any $z = (x, t) \in \mathcal{M}_0 \times \mathbb{R}$ and $\zeta = (\xi, \tau) \in T_z\mathcal{M} \equiv T_x\mathcal{M}_0 \times \mathbb{R}$, with $\beta: \mathcal{M}_0 \to \mathbb{R}$ smooth and positive scalar field, $\delta: \mathcal{M}_0 \to T\mathcal{M}_0$ smooth vector field. In particular, if $\delta(x) \equiv 0$, (1.1) defines a static metric and $(\mathcal{M}, \langle \cdot, \cdot \rangle_z)$ is called static Lorentzian manifold.

From now on, let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a stationary Lorentz manifold equipped with the Lorentz metric (1.1). Let P_0 and P_1 be two given submanifolds of $\mathcal{M}_0$ and let $t_0 \in \mathbb{R}$ be fixed.

The aim of this paper is to study the existence of lightlike geodesics $z : [0, 1] \to \mathcal{M}$, $z = (x, t)$, joining $\tilde{P}_0 = P_0 \times \{t_0\}$ and $\tilde{P}_1 = P_1 \times \mathbb{R}$, that is such that $z(0) \in \tilde{P}_0$ and $z(1) \in \tilde{P}_1$, and whose space component x satisfies the orthogonal conditions.

$$\begin{cases} \langle \dot{x}(0), \xi \rangle_x = 0 & \text{for all } \xi \in T_{x(0)} P_0, \\ \langle \dot{x}(1), \xi \rangle_x = 0 & \text{for all } \xi \in T_{x(1)} P_1. \end{cases} \tag{1.2}$$

More exactly we want to find smooth functions $z : [0, 1] \to \mathcal{M}$, $z(s) = (x(s), t(s))$, solutions of the following system:

$$\begin{cases} D_s \dot{z}(s) = 0 & \text{for all } s \in [0, 1], \\ E(z) = \langle \dot{z}(s), \dot{z}(s) \rangle_z = 0 & \text{for all } s \in [0, 1], \\ x(0) \in P_0, \quad t(0) = t_0, \quad x(1) \in P_1, & \\ \langle \dot{x}(0), \xi \rangle_x = 0 & \text{for all } \xi \in T_{x(0)} P_0, \\ \langle \dot{x}(1), \xi \rangle_x = 0 & \text{for all } \xi \in T_{x(1)} P_1. \end{cases} \tag{1.3}$$

From a physical point of view, a Lorentz metric describes a gravitational field and lightlike geodesics verifying (1.3) represent trajectories of light rays joining two celestial bodies of which one is a light source. In General Relativity a remarkable example of stationary Lorentz manifold is the Kerr space–time which describes the space–time outside an axially symmetric body rotating around its axis while an example of static manifold is the Schwarzschild space–time which represents the manifold outside a static spherically symmetric massive body (cf. [6,9]).

Lightlike geodesics joining an event $\tilde{P}_0 = \{(x_0, t_0)\}$ to a vertical line $\tilde{P}_1 = \{x_1\} \times \mathbb{R}$ have been studied in [5], while in [12] the existence of geodesics, not necessarily lightlike, from a point to a subspace $\tilde{P}_1 = P_1 \times \{t_1\}$ in a static Lorentzian manifold has been proved. Here we prove that, in a stationary manifold $\mathcal{M}$, the number of light rays joining $\tilde{P}_0 = P_0 \times \{t_0\}$ and $\tilde{P}_1 = P_1 \times \mathbb{R}$ depends on the topological properties of $\mathcal{M}_0$, P_0 and P_1 (for the Riemannian case, see [8,13]). To this aim in Section 3 we find out a lower bound to the Ljusternik–Schnirelman category of the space of paths joining P_0 to P_1 in $\mathcal{M}_0$ by means of the category of $P_0 \times P_1$.

In the following, the Ljusternik–Schnirelman category of the topological space X in itself will be denoted by $\mathrm{cat}(X)$ (see Definition 3.1).

Theorem 1.1. *Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a manifold equipped with the stationary Lorentz metric* (1.1) *such that*
(M_0) *$(\mathcal{M}_0, \langle\cdot,\cdot\rangle_x)$ is a connected, complete, C^3 n-dimensional Riemannian manifold;*
(M_1) *there exist some constants ν, N, $D > 0$ such that*

$$\nu \le \beta(x) \le N \quad \text{and} \quad \langle\delta(x), \delta(x)\rangle_x \le D \quad \text{for all } x \in \mathcal{M}_0. \tag{1.4}$$

Let P_0 and P_1 be two disjoint closed submanifolds of $\mathcal{M}_0$ such that
(C) *P_0 or P_1 is compact;*
(O_0) *$\langle\delta(x), \xi\rangle_x = 0$ for any $x \in P_0$, $\xi \in T_x P_0$;*
(O_1) *$\langle\delta(x), \xi\rangle_x = 0$ for any $x \in P_1$, $\xi \in T_x P_1$.*
Then there exists at least one solution of (1.3). *If, moreover, P_0 and P_1 are both contractible in $\mathcal{M}_0$, then problem* (1.3) *has at least* $\mathrm{cat}(P_0 \times P_1)$ *solutions.*

The following multiplicity theorem holds even if, eventually, P_0 and P_1 are not disjoint:

Theorem 1.2. *Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a manifold equipped with the Lorentz metric* (1.1) *such that hypotheses* (M_0) *and* (M_1) *are satisfied. Let P_0 and P_1 be two closed submanifolds of $\mathcal{M}_0$ such that* (C), (O_0) *and* (O_1) *hold. If $\mathcal{M}_0$ is not contractible in itself while P_0 and P_1 are both contractible in $\mathcal{M}_0$, then problem* (1.3) *has infinitely many non-constant solutions $z_n(s) = (x_n(s), t_n(s))$ whose "arrival times" $(t_n(1))_{n\in\mathbb{N}}$ form a diverging increasing sequence.*

Remark 1.3. Let $(\mathcal{M}, \langle\cdot,\cdot\rangle_z)$ be a conformal stationary Lorentz manifold, that is there exists a finite-dimensional Riemannian manifold $(\mathcal{M}_0, \langle\cdot,\cdot\rangle_x)$ such that $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ is equipped with the Lorentz metric

$$\langle\zeta, \zeta\rangle_z = \alpha(x,t)\,[\langle\xi,\xi\rangle_x + 2\langle\delta(x), \xi\rangle_x \tau - \beta(x)\tau^2]$$

for any $z = (x,t) \in \mathcal{M}_0 \times \mathbb{R}$ and for any $\zeta = (\xi, \tau) \in T_z\mathcal{M} \equiv T_x\mathcal{M}_0 \times \mathbb{R}$, where $\alpha: \mathcal{M} \to \mathbb{R}$ and $\beta: \mathcal{M}_0 \to \mathbb{R}$ are smooth and positive scalar fields and $\delta: \mathcal{M}_0 \to T\mathcal{M}_0$ is a smooth vector field. Observe that β may not satisfy (1.4). Let P_0, P_1 be two submanifolds of $\mathcal{M}_0$ and let $t_0 \in \mathbb{R}$ be fixed. Since lightlike geodesics are independent, up to reparametrization, on a conformal change of the metric, the same results of Theorems 1.1 and 1.2 still hold for such a kind of Lorentz manifolds provided that in the hypotheses of such theorems we replace the Riemannian metric $\langle\cdot,\cdot\rangle_x$ on $T_x\mathcal{M}_0$ with the new one

$$\langle\cdot,\cdot\rangle_R = \frac{\langle\cdot,\cdot\rangle_x}{\beta(x)} \quad \text{for each } x \in \mathcal{M}_0.$$

Remark 1.4. Clearly, Theorems 1.1 and 1.2 can be proved if $\delta(x) \equiv 0$, that is if $\mathcal{M}$ is a static Lorentz manifold. In this case, however, the proof can be given by a different and easier variational approach (see Section 5).

If P_0 is reduced to a single point $x_0 \in \mathcal{M}_0$ the existence and multiplicity results in [12] can be improved as follows.

Corollary 1.5. *Let* $(\mathcal{M}, \langle\cdot,\cdot\rangle_z)$ *be a stationary Lorentz manifold satisfying* (M_0) *and* (M_1). *Let* $z_0 = (x_0, t_0) \in \mathcal{M}$ *be fixed and* P_1 *be a closed submanifold of* $\mathcal{M}_0$ *such that* (O_1) *holds. If* $x_0 \notin P_1$, *there exists at least one lightlike geodesic starting from* z_0 *and ending in* $P_1 \times \mathbb{R}$; *moreover if* P_1 *is contractible in* $\mathcal{M}_0$ *such geodesics are at least* $\mathrm{cat}(P_1)$. *At last, either if* $x_0 \in P_1$ *or if* $x_0 \notin P_1$, *if* P_1 *is contractible in* $\mathcal{M}_0$ *while* $\mathcal{M}_0$ *is not contractible in itself, there exist infinitely many lightlike non-constant geodesics* $z_n(s) = (x_n(s), t_n(s))$ *from* z_0 *to* $P_1 \times \mathbb{R}$ *whose "arrival times"* $(t_n(1))_{n\in\mathbb{N}}$ *form a diverging increasing sequence.*

Remark 1.6. By using the arguments in [2,13] it is possible to extend Theorems 1.1 and 1.2 to the case in which the Lorentzian manifold has a light convex boundary or the submanifolds P_0 and P_1 are both non-compact. In particular, this generalization allows to prove the existence of light rays of type (1.3) in some non-complete space–times relevant from a physical point of view, for example the Kerr and the Schwarzschild ones.

2. Variational approach

Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a manifold equipped with the stationary Lorentz metric (1.1) such that hypotheses (M_0) and (M_1) hold. Let $t_0 \in \mathbb{R}$ be fixed and P_0, P_1 be two given submanifolds of $\mathcal{M}_0$. Assume $t_0 = 0$ and let $\tilde{P}_0 = P_0 \times \{0\}$, $\tilde{P}_1 = P_1 \times \mathbb{R}$.

As lightlike geodesics are independent, up to reparametrization, on a conformal change of the metric and β is bounded and far from zero, then, without loss of generality, we can assume that it is $\beta(x) \equiv 1$; moreover by the Nash Embedding theorem it follows that $\mathcal{M}_0$ is a submanifold of an Euclidean space $\mathbb{R}^N$ and its metric $\langle\cdot,\cdot\rangle_x$ is the Euclidean metric of $\mathbb{R}^N$ which will be denoted by $\langle\cdot,\cdot\rangle$, thus we can suppose that $\mathcal{M}$ is equipped with the Lorentz metric

$$\langle\zeta,\zeta\rangle_z = \langle\xi,\xi\rangle + 2\,\langle\delta(x),\xi\rangle\,\tau - \tau^2 \tag{2.1}$$

for any $z = (x, t) \in \mathcal{M}_0 \times \mathbb{R}$ and $\zeta = (\xi, \tau) \in T_z\mathcal{M} \equiv T_x\mathcal{M}_0 \times \mathbb{R}$.

Let $I = [0, 1]$ and $H^1(I, \mathbb{R}^N)$ be the Sobolev space of the absolutely continuous curves whose derivative is square summable. It is well known that $H^1(I, \mathbb{R}^N)$ is a Hilbert space endowed by the norm

$$\|x\|^2 = \int_0^1 \langle\dot{x},\dot{x}\rangle\,\mathrm{d}s + \int_0^1 \langle x, x\rangle\,\mathrm{d}s.$$

Let us define the subset

$$\Gamma(P_0, P_1) = \{x \in H^1(I, \mathbb{R}^N) : x(I) \subset \mathcal{M}_0\ ;\ x(0) \in P_0,\ x(1) \in P_1\}.$$

It is possible to prove (see, e.g., [10]) that if $\mathcal{M}_0$ is complete and P_0, P_1 are closed then $\Gamma(P_0, P_1)$ is a complete Riemannian manifold whose tangent space in $x \in \Gamma(P_0, P_1)$ is

$$T_x\Gamma(P_0, P_1) = \{\xi \in H^1(I, T\mathcal{M}_0) :\ \xi(s) \in T_{x(s)}\mathcal{M}_0 \text{ for all } s \in I\ ;\ \xi(0) \in T_{x(0)}P_0,\ \xi(1) \in T_{x(1)}P_1\}.$$

If (O_0) and (O_1) hold, then solutions of (1.3) can be found as critical points at level zero of the functional

$$f(z) = \frac{1}{2} \int_0^1 \langle \dot{z}, \dot{z} \rangle_z \, ds$$

in $\Gamma(P_0, P_1) \times H^1(I, \mathbb{R})$. Unluckly if f is not bounded from above nor from below then, as in [5], it is better to define a new functional bounded from below by introducing a new parameter, the "arrival time" $\lambda \in \mathbb{R}$, and a variational argument similar to the Fermat principle.

Fixed $\lambda \in \mathbb{R}$, let us introduce

$$W_\lambda = \{t \in H^1(I, \mathbb{R}): t(0) = 0, \ t(1) = \lambda\},$$

closed affine submanifold of $H^1(I, \mathbb{R})$ whose tangent space in each point is given by

$$H_0^1 = \{\tau \in H^1(I, \mathbb{R}): \tau(0) = \tau(1) = 0\},$$

and let us define $Z_\lambda = \Gamma(P_0, P_1) \times W_\lambda$ Hilbert manifold such that $T_z Z_\lambda \equiv T_x \Gamma(P_0, P_1) \times H_0^1$ for each $z = (x, t) \in Z_\lambda$.

Let us consider the "energy" functional restricted to Z_λ

$$f_\lambda(z) = \frac{1}{2} \int_0^1 \langle \dot{z}, \dot{z} \rangle_z \, ds = \frac{1}{2} \int_0^1 (\langle \dot{x}, \dot{x} \rangle + 2 \langle \delta(x), \dot{x} \rangle \dot{t} - \dot{t}^2) \, ds, \quad z = (x, t) \in Z_\lambda.$$

Remark 2.1. It is easy to prove that f_λ is a C^1 functional on Z_λ; moreover if $z = (x, t) \in Z_\lambda$ and $\zeta = (\xi, \tau) \in T_z Z_\lambda$, by $\tau \in H_0^1$ and integrating by parts there results

$$\begin{aligned} f'_\lambda(z)[\zeta] = & \int_0^1 \langle \dot{z}, \dot{\zeta} \rangle_z \, ds = \langle \dot{z}(1), \zeta(1) \rangle_z - \langle \dot{z}(0), \zeta(0) \rangle_z - \int_0^1 \langle D_s \dot{z}, \zeta \rangle_z \, ds \\ = & \int_0^1 \left\langle - D_s \dot{x} + \dot{t} \, \delta'(x)^*[\dot{x}] - \frac{d}{ds}(\dot{t}\delta(x)), \xi \right\rangle ds + [\langle \dot{x}, \xi \rangle]_0^1 \\ & + [\dot{t} \, \langle \delta(x), \xi \rangle]_0^1 + \int_0^1 \left(\ddot{t} - \frac{d}{ds} (\langle \delta(x), \dot{x} \rangle) \right) \tau \, ds, \end{aligned}$$

where $\delta'(x(s))^*$ is the adjoint of $\delta'(x(s))$ for any $s \in I$. Clearly,

$$\begin{aligned} \frac{\partial f_\lambda}{\partial x}(z)[\xi] = & f'_\lambda(z)[(\xi, 0)] \\ = & \int_0^1 \left\langle - D_s \dot{x} + \dot{t} \, \delta'(x)^*[\dot{x}] - \frac{d}{ds}(\dot{t}\delta(x)), \xi \right\rangle ds \\ & + [\langle \dot{x}, \xi \rangle]_0^1 + [\dot{t} \, \langle \delta(x), \xi \rangle]_0^1 \end{aligned} \tag{2.2}$$

for all $\xi \in T_x\Gamma(P_0, P_1)$, while

$$\frac{\partial f_\lambda}{\partial t}(z)[\tau] = f'_\lambda(z)[(0,\tau)] = \int_0^1 \left(\ddot{t} - \frac{\mathrm{d}}{\mathrm{d}s}(\langle\delta(x), \dot{x}\rangle)\right)\tau \,\mathrm{d}s \tag{2.3}$$

for all $\tau \in H_0^1$.

Theorem 2.2. *Let $z: s \in I \longmapsto z(s) = (x(s), t(s)) \in \mathcal{M}$. If P_0 and P_1 satisfy the orthogonal hypotheses* (O_0) *and* (O_1), *then the following propositions are equivalent:*
(a) *z is a solution of* (1.3) *with "arrival time" $t(1) = \lambda$;*
(b) *z is a critical point of f_λ on Z_λ such that $f_\lambda(z) = 0$.*

Proof. Remark that conditions (O_0) and (O_1) imply

$$\langle\delta(x(0)), \xi(0)\rangle = \langle\delta(x(1)), \xi(1)\rangle = 0 \quad \text{for all } \xi \in T_x\Gamma(P_0, P_1). \tag{2.4}$$

If (a) holds, then (b) follows easily by Remark 2.1, (2.4) and the orthogonal conditions (1.2).

Let z be such that $f'_\lambda(z) = 0$. By (2.3) it follows

$$\ddot{t} - \frac{\mathrm{d}}{\mathrm{d}s}(\langle\delta(x), \dot{x}\rangle) = 0\,; \tag{2.5}$$

moreover by (2.2) for any $\xi \in T_x\Gamma(P_0, P_1)$ with compact support it is

$$\int_0^1 \left\langle -D_s\dot{x} + \dot{t}\,\delta'(x)^*[\dot{x}] - \frac{\mathrm{d}}{\mathrm{d}s}(\dot{t}\delta(x)), \xi\right\rangle \mathrm{d}s = 0.$$

By using classical theorems it can be proved that

$$-D_s\dot{x} + \dot{t}\,\delta'(x)^*[\dot{x}] - \frac{\mathrm{d}}{\mathrm{d}s}(\dot{t}\delta(x)) = 0,$$

then (2.4) implies that z is a geodesic and the orthogonal conditions (1.2) hold, while $f_\lambda(z) = 0$ implies that z is lightlike. □

From now on, let P_0 and P_1 satisfy the orthogonal hypotheses (O_0) and (O_1).

Let us consider the kernel of the map $\partial f_\lambda/\partial t$:

$$N_\lambda = \left\{z \in Z_\lambda : \frac{\partial f_\lambda}{\partial t}(z) \equiv 0\right\}.$$

Proposition 2.3. *Let $z = (x, t) \in Z_\lambda$ be given. Then the following propositions are equivalent:*
(a) *z is a critical point of f_λ;*
(b) *$z \in N_\lambda$ and*

$$\frac{\partial f_\lambda}{\partial x}(z)[\xi] = 0 \quad \textit{forall } \xi \in T_x\Gamma(P_0, P_1).$$

Proof. Follows easily by Remark 2.1. □

Remark 2.4. Let $z = (x, t) \in Z_\lambda$. By (2.5) it follows that $z \in N_\lambda$ if and only if

$$t(s) = \int_0^s \langle \delta(x(r)), \dot{x}(r) \rangle \, \mathrm{d}r \; + \; s \left(\lambda - \int_0^1 \langle \delta(x), \dot{x} \rangle \, \mathrm{d}r \right) \quad \text{for all } s \in I.$$

Let us define

$$\Phi_\lambda : x \in \Gamma(P_0, P_1) \longmapsto \Phi_\lambda(x) \in W_\lambda$$

such that

$$\Phi_\lambda(x)(s) = \int_0^s \langle \delta(x(r)), \dot{x}(r) \rangle \, \mathrm{d}r + s \left(\lambda - \int_0^1 \langle \delta(x), \dot{x} \rangle \, \mathrm{d}r \right) \text{ for all } s \in I.$$

By Remark 2.4 it is easy to prove that Φ_λ is a C^1 function whose graph is just N_λ, that is

$$z = (x, t) \in N_\lambda \quad \Longleftrightarrow \quad t = \Phi_\lambda(x). \tag{2.6}$$

By (2.6) it follows that the restriction of f_λ on N_λ is the functional

$$J_\lambda : x \in \Gamma(P_0, P_1) \longmapsto J_\lambda(x) = f_\lambda(x, \Phi_\lambda(x)) \in \mathbb{R},$$

hence for each $x \in \Gamma(P_0, P_1)$:

$$J_\lambda(x) = \frac{1}{2} \int_0^1 (\langle \dot{x}, \dot{x} \rangle + \langle \delta(x), \dot{x} \rangle^2) \, \mathrm{d}s \; - \left(\lambda - \int_0^1 \langle \delta(x), \dot{x} \rangle \, \mathrm{d}s \right)^2 . \tag{2.7}$$

Let us remark that

$$J'_\lambda(x)[\xi] = \frac{\partial f_\lambda}{\partial x}(x, \Phi_\lambda(x))[\xi] \; + \; \frac{\partial f_\lambda}{\partial t}(x, \Phi_\lambda(x)) \left[\Phi'_\lambda(x)[\xi] \right], \tag{2.8}$$

for any $x \in \Gamma(P_0, P_1)$, $\xi \in T_x\Gamma(P_0, P_1)$.

Arguing as in [7], Proposition 2.3, (2.6) and (2.8) imply the following result:

Proposition 2.5. *Taken $z = (x, t) \in Z_\lambda$, the following propositions are equivalent:*
(a) *z is a critical point of f_λ;*
(b) *x is a critical point of J_λ and $t = \Phi_\lambda(x)$.*
Moreover, if (a) *or* (b) *holds, it is $f_\lambda(x, t) = J_\lambda(x)$.*

If $\lambda \in \mathbb{R}$ is fixed, by Theorem 2.2 and Proposition 2.5 it follows that, for obtaining solutions of problem (1.3) such that $t(1) = \lambda$, it is enough to find critical points of J_λ such that $J_\lambda(x) = 0$. Unluckly here λ is unknown and, as it gives the "instant" in which the lightlike geodesic z "arrives" to the given manifold $\tilde{P}_1$, we can suppose that the parameter λ has to be strictly positive.

Let us introduce the map

$$H\colon (\lambda, x) \in \mathbb{R}_+ \times \Gamma(P_0, P_1) \longmapsto 2J_\lambda(x) \in \mathbb{R}.$$

Remark 2.6. By properties of J_λ and by last remarks it follows that H is a C^1 functional and solving (1.3) is equivalent to find $(\lambda, x) \in \mathbb{R}_+ \times \Gamma(P_0, P_1)$ solution of the following problem:

$$\frac{\partial H}{\partial x}(\lambda, x) = 0, \qquad H(\lambda, x) = 0, \qquad \lambda > 0. \tag{2.9}$$

Remark 2.7. By (2.7) it is easy to prove that if (λ, x) is such that $H(\lambda, x) = 0$, then

$$\frac{\partial H}{\partial \lambda}(\lambda, x) = -2\left(\lambda - \int_0^1 \langle \delta(x), \dot{x}\rangle \,\mathrm{d}s\right) = 0 \quad \Longrightarrow \quad x \text{ is constant.}$$

Let $F\colon \Gamma(P_0, P_1) \to \mathbb{R}$ be defined as follows:

$$F(x) = \int_0^1 \langle \delta(x), \dot{x}\rangle \,\mathrm{d}s + \sqrt{\int_0^1 (\langle \dot{x}, \dot{x}\rangle + \langle \delta(x), \dot{x}\rangle^2) \,\mathrm{d}s}. \tag{2.10}$$

Remark 2.8. As Hölder older inequality implies

$$\left(\int_0^1 \langle \delta(x), \dot{x}\rangle \,\mathrm{d}s\right)^2 \le \int_0^1 \langle \delta(x), \dot{x}\rangle^2 \,\mathrm{d}s, \tag{2.11}$$

by (1.4) and (2.10) arguing as in [2, Lemma 3.2], it is possible to prove that there exists $c_0 > 0$ such that

$$F(x) \ge c_0 \left(\int_0^1 \langle \dot{x}, \dot{x}\rangle \,\mathrm{d}s\right)^{\frac{1}{2}} \quad \text{for all } x \in \Gamma(P_0, P_1). \tag{2.12}$$

Whence $F(x) \ge 0$ for each $x \in \Gamma(P_0, P_1)$ while $F(x) = 0$ if and only if x is a constant function.

By simple calculations it is possible to prove that F is a map continuous but not differentiable at level zero and it is smooth elsewhere.

Remark 2.9. By Remarks 2.7, 2.8 and (2.10) it can be proved (see, e.g. [2] or [5]) that $(\bar{\lambda}, \bar{x})$ solves (2.9) with $\bar{x}$ non-constant if and only if $\bar{x}$ is such that

$$F'(\bar{x}) = 0, \quad \bar{\lambda} = F(\bar{x}) > 0.$$

Remark 2.10. If the given closed submanifolds P_0 and P_1 are disjoint, then there are no constants in $\Gamma(P_0, P_1)$; hence F is a C^1 strictly positive functional in $\Gamma(P_0, P_1)$ and by Remark 2.9 it follows that for solving (2.9) it is enough to find critical points of F.

Finally, by Remarks 2.6 and 2.9 it follows:

Theorem 2.11. *Let P_0 and P_1 be two given submanifolds of $\mathcal{M}_0$ which satisfy the orthogonal conditions* (O_0) *and* (O_1). *If $\bar{x} \in \Gamma(P_0, P_1)$ is such that*

$$F'(\bar{x}) = 0, \qquad F(\bar{x}) > 0,$$

then, assuming $\bar{\lambda} = F(\bar{x})$, $\bar{z} = (\bar{x}, \Phi_{\bar{\lambda}}(\bar{x}))$ is a solution of problem (1.3).

3. Topological tools

In the last paragraph it has been proved that solving the given problem is equivalent to find positive critical levels of the functional F defined in (2.10). To this aim we will use the well known Ljusternik–Schnirelman Theory (see, e.g. [11,14,15]).

Definition 3.1. Let X be a topological space. Given $A \subseteq X$, $\mathrm{cat}_X(A)$ is the category of A in X, that is the least number of closed and contractible subsets of X covering A. If it is not possible to cover A with a finite number of such sets, it is $\mathrm{cat}_X(A) = +\infty$.

We denote $\mathrm{cat}(X) = \mathrm{cat}_X(X)$.

Definition 3.2. Let Γ be a Riemannian manifold. A C^1 functional $g: \Gamma \to \mathbb{R}$ satisfies the Palais–Smale condition at level $a \in \mathbb{R}$, briefly $(\mathrm{PS})_a$, if any $(x_n)_{n\in\mathbb{N}} \subset \Gamma$ such that $g(x_n) \to a$ and $g'(x_n) \to 0$ for $n \to +\infty$ has a subsequence which converges in Γ.

Let us recall the classical Ljusternik–Schnirelman multiplicity theorem.

Theorem 3.3. *Let Γ be a complete Riemannian manifold and g a C^1 functional on Γ which satisfies the $(PS)_a$ condition at any level $a \in \mathbb{R}$. Taking any $k \in \mathbb{N}$, $k > 0$, let us define*

$$\Gamma_k = \{A \subseteq \Gamma: \mathrm{cat}_\Gamma(A) \geq k\}, \qquad c_k = \inf_{A\in\Gamma_k} \sup_{x\in A} g(x). \tag{3.1}$$

Then c_k is a critical value of g for each k such that $\Gamma_k \neq \emptyset$ and $c_k \in \mathbb{R}$; if, moreover, g is bounded from below then g attains its infimum and has at least $\mathrm{cat}(\Gamma)$ *critical levels.*

Remark 3.4. Let Γ and g be as in Theorem 3.3. If g is bounded from below then for all $c \in \mathbb{R}$ it is

$$\mathrm{cat}_\Gamma(g^c) < +\infty,$$

where $g^c = \{x \in \Gamma: g(x) \leq c\}$ is the sublevel of g corresponding to the level c.

Remark 3.5. If g is a positive functional not differentiable at level zero while it is smooth elsewhere in a complete Riemannian manifold and the $(\mathrm{PS})_a$ condition holds at any level $a > 0$, then it can be proved that the same result of Theorem 3.3 holds for $c_k > 0$.

As Theorem 3.3 joins critical levels of a functional g to the topology of Γ, let us give some theorems useful to know more about the topological properties of the manifold $\Gamma(P_0, P_1)$ introduced in the previous section.

Theorem 3.6. *Let $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ be a smooth complete connected finite-dimensional Riemannian manifold and P_0 and P_1 be closed submanifolds. If $\mathcal{M}_0$ is not contractible in itself while both P_0 and P_1 are contractible in $\mathcal{M}_0$, then $\Gamma(P_0, P_1)$ has infinite category and possesses compact subsets of arbitrary high category.*

Proof. For the proof, see [3,4].

Theorem 3.7. *Let $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ be a smooth complete connected finite-dimensional Riemannian manifold and P_0 and P_1 be closed submanifolds both contractible in $\mathcal{M}_0$. Then*

$$\mathrm{cat}(\Gamma(P_0, P_1)) \geq \mathrm{cat}(P_0 \times P_1).$$

Proof. If $\mathcal{P}(P_0, P_1)$ denotes the space of paths in $\mathcal{M}_0$ which start from P_0 and end in P_1, by $\mathrm{cat}(\mathcal{P}(P_0, P_1)) = \mathrm{cat}(\Gamma(P_0, P_1))$ (e.g., see [4]) it is enough to prove that

$$\mathrm{cat}(\mathcal{P}(P_0, P_1)) \geq \mathrm{cat}(P_0 \times P_1). \tag{3.2}$$

Let $m = \mathrm{cat}(\mathcal{P}(P_0, P_1))$ and $A_1, A_2, \ldots, A_m$ be closed and contractible sets in $\mathcal{P}(P_0, P_1)$ such that $\mathcal{P}(P_0, P_1) = A_1 \cup A_2 \cup \cdots \cup A_m$.

For any $j \in \{1, 2, \ldots, m\}$, define

$$B_j = \{(q_0, q_1) \in P_0 \times P_1 : \text{ there exists } x \in A_j \text{ such that } x(0) = q_0,\ x(1) = q_1\}.$$

By Definition 3.1 it follows that (3.2) holds if

$$P_0 \times P_1 = \bigcup_{j=1}^{m} B_j \tag{3.3}$$

and B_j is closed and contractible in $P_0 \times P_1$ for each $j \in \{1, 2, \ldots, m\}$.

Remark that as P_0 and P_1 are contractible in $\mathcal{M}_0$, then there exist $\bar{q}_0, \bar{q}_1 \in \mathcal{M}_0$ and two continuous maps $H_0 : I \times P_0 \to \mathcal{M}_0$, $H_1 : I \times P_1 \to \mathcal{M}_0$ such that

$$H_0(0, q_0) = q_0, \qquad H_0(1, q_0) = \bar{q}_0 \qquad \text{for all } q_0 \in P_0, \tag{3.4}$$

$$H_1(0, q_1) = q_1, \qquad H_1(1, q_1) = \bar{q}_1 \qquad \text{for all } q_1 \in P_1\ ; \tag{3.5}$$

moreover $\mathcal{M}_0$ connected implies that there exists a path $\alpha: I \to \mathcal{M}_0$ such that

$$\alpha(0) = \bar{q}_0, \qquad \alpha(1) = \bar{q}_1. \tag{3.6}$$

Let $(q_0, q_1) \in P_0 \times P_1$ be fixed and assume $\omega_{q_0,q_1}: I \to \mathcal{M}_0$ as follows:

$$\omega_{q_0,q_1}(s) = \begin{cases} H_0(3s, q_0) & \text{if } s \in [0, \frac{1}{3}[, \\ \alpha(3s-1) & \text{if } s \in [\frac{1}{3}, \frac{2}{3}], \\ H_1(3-3s, q_1) & \text{if } s \in]\frac{2}{3}, 1]. \end{cases} \tag{3.7}$$

By (3.4), (3.5) and (3.6) it follows that

$$\omega_{q_0,q_1} \in \mathcal{P}(P_0, P_1), \qquad \omega_{q_0,q_1}(0) = q_0, \qquad \omega_{q_0,q_1}(1) = q_1, \tag{3.8}$$

thus there exists $j \in \{1, 2, \dots, m\}$ such that $\omega_{q_0,q_1} \in A_j$, hence $(q_0, q_1) \in B_j$ and (3.3) holds.

Now, let $j \in \{1, 2, \dots, m\}$ be fixed. We claim that B_j is closed and contractible in $P_0 \times P_1$.

In fact, let $(q_{0n}, q_{1n}) \in B_j$ be a given sequence such that $(q_{0n}, q_{1n}) \to (q_0, q_1)$ in $\mathcal{M}_0 \times \mathcal{M}_0$ if $n \to +\infty$. By P_0 and P_1 closed, it is $(q_0, q_1) \in P_0 \times P_1$ and, defined $\omega_{q_{0n},q_{1n}}$ and ω_{q_0,q_1} as in (3.7), it is easy to prove that $\omega_{q_{0n},q_{1n}} \to \omega_{q_0,q_1}$ uniformly in $\mathcal{P}(P_0, P_1)$. As $\omega_{q_{0n},q_{1n}} \in A_j$ and A_j is closed, then $\omega_{q_0,q_1} \in A_j$ whence by (3.8) it follows $(q_0, q_1) \in B_j$.

As A_j is contractible in $\mathcal{P}(P_0, P_1)$, there exist $\bar{x}_j \in \mathcal{P}(P_0, P_1)$ and a continuous map $\mathcal{H}_j : I \times A_j \to \mathcal{P}(P_0, P_1)$ such that

$$\mathcal{H}_j(0, x) = x, \qquad \mathcal{H}_j(1, x) = \bar{x}_j \quad \text{for all } x \in A_j. \tag{3.9}$$

For $s \in I$ and $(q_0, q_1) \in B_j$, assume

$$\tilde{\mathcal{H}}_j(s, (q_0, q_1)) = (\mathcal{H}_j(s, \omega_{q_0,q_1})(0), \mathcal{H}_j(s, \omega_{q_0,q_1})(1))$$

with ω_{q_0,q_1} defined in (3.7). It is easy to prove that $\tilde{\mathcal{H}}_j : I \times B_j \to P_0 \times P_1$ is continuous, moreover (3.8) and (3.9) imply that

$$\begin{aligned} \tilde{\mathcal{H}}_j(0, (q_0, q_1)) &= (\mathcal{H}_j(0, \omega_{q_0,q_1})(0), \mathcal{H}_j(0, \omega_{q_0,q_1})(1)) \\ &= (\omega_{q_0,q_1}(0), \omega_{q_0,q_1}(1)) = (q_0, q_1), \end{aligned}$$

while

$$\tilde{\mathcal{H}}_j(1, (q_0, q_1)) = (\mathcal{H}_j(1, \omega_{q_0,q_1})(0), \mathcal{H}_j(1, \omega_{q_0,q_1})(1)) = (\bar{x}_j(0), \bar{x}_j(1)),$$

for all $(q_0, q_1) \in B_j$. Hence B_j is contractible to $(\bar{x}_j(0), \bar{x}_j(1))$ in $P_0 \times P_1$. □

Remark 3.8. Theorem 3.7 generalizes a similar result proved in [12] when $P_0 = \{x_0\}$.

4. Proof of main theorems

From now on, let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a manifold equipped with the Lorentz metric (2.1) such that hypotheses (M_0) and (M_1) are satisfied. Moreover, let P_0 and P_1 be two given closed submanifolds of $\mathcal{M}_0$ such that (C), (O_0) and (O_1) hold.

In order to find positive critical levels of the functional F introduced in (2.10), we need the following lemma.

Lemma 4.1. *The functional F satisfies the $(PS)_a$ condition at any strictly positive level a.*

Proof. Let $a > 0$ and $(x_n)_{n\in\mathbb{N}} \subset \Gamma(P_0, P_1)$ be a $(\mathrm{PS})_a$ sequence for F, i.e.

$$\lim_{n\to+\infty} F(x_n) = a, \tag{4.1}$$

$$\lim_{n\to+\infty} F'(x_n) = 0. \tag{4.2}$$

By (2.12) and (4.1) it follows that

$$\left(\int_0^1 \langle \dot{x}_n, \dot{x}_n\rangle \,\mathrm{d}s\right)_{n\in\mathbb{N}} \qquad \text{is bounded.} \tag{4.3}$$

By the hypothesis (C) and (4.3) it is easy to prove that there exists $x_0 \in \mathcal{M}_0$ such that

$$\sup\{d(x_n(s), x_0) : s \in I,\ n \in \mathbb{N}\} < +\infty, \tag{4.4}$$

(where $d(\cdot, \cdot)$ is the distance in $\mathcal{M}_0$) and $(x_n)_{n\in\mathbb{N}}$ is bounded in $H^1(I, \mathbb{R}^N)$, so there exists x such that $x_n \rightharpoonup x$ weakly in $H^1(I, \mathbb{R}^N)$ and uniformly in I, up to subsequences.

As $\mathcal{M}_0$ is complete and P_0, P_1 are closed, then $x \in \Gamma(P_0, P_1)$; moreover it is possible to prove (cf. [1, Lemma 2.1]) that there exist two bounded sequences $(\xi_n)_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}}$ in $H^1(I, \mathbb{R}^N)$ such that

$$x_n - x = \xi_n + \nu_n, \quad \xi_n \in T_{x_n}\Gamma(P_0, P_1) \quad \text{for any } n \in \mathbb{N}, \tag{4.5}$$

$$\xi_n \rightharpoonup 0 \text{ weakly in } H^1(I, \mathbb{R}^N) \quad \text{and} \quad \nu_n \to 0 \text{ strongly in } H^1(I, \mathbb{R}^N). \tag{4.6}$$

By (4.2) and (4.6) it follows that

$$\mathrm{o}(1) = F'(x_n)[\xi_n] = \int_0^1 \left(\langle \delta'(x_n)\xi_n, \dot{x}_n\rangle + \langle \delta(x_n), \dot{\xi}_n\rangle\right) \mathrm{d}s$$
$$+ \frac{\int_0^1 (\langle \dot{x}_n, \dot{\xi}_n\rangle + \langle \delta(x_n), \dot{x}_n\rangle\, \langle \delta(x_n), \dot{\xi}_n\rangle + \langle \delta(x_n), \dot{x}_n\rangle\, \langle \delta'(x_n)\xi_n, \dot{x}_n\rangle)\, \mathrm{d}s}{\sqrt{\int_0^1 (\langle \dot{x}_n, \dot{x}_n\rangle + \langle \delta(x_n), \dot{x}_n\rangle^2)\, \mathrm{d}s}}.$$

By (4.4) and $x_n \to x$ uniformly in I it follows $\delta(x_n) \to \delta(x)$ uniformly in I, thus by (4.6) it is

$$\int_0^1 \langle \delta(x_n), \dot{\xi}_n\rangle \,\mathrm{d}s = \int_0^1 \langle \delta(x_n) - \delta(x), \dot{\xi}_n\rangle \,\mathrm{d}s + \int_0^1 \langle \delta(x), \dot{\xi}_n\rangle \,\mathrm{d}s = \mathrm{o}(1). \tag{4.7}$$

By simple calculations, (1.4) and (4.3) imply

$$\left(\sqrt{\int_0^1 (\langle \dot{x}_n, \dot{x}_n\rangle + \langle \delta(x_n), \dot{x}_n\rangle^2)\,\mathrm{d}s}\right)_{n\in\mathbb{N}} \quad \text{is bounded,}$$

then by (4.3), (4.4), (4.6) and (4.7) it follows

$$\int_0^1 \langle \dot{x}_n, \dot{\xi}_n\rangle\,\mathrm{d}s + \int_0^1 \langle \delta(x_n), \dot{x}_n\rangle\langle \delta(x_n), \dot{\xi}_n\rangle\,\mathrm{d}s = \mathrm{o}(1). \tag{4.8}$$

By means of (4.5)–(4.7), Eq. (4.8) becomes

$$\int_0^1 \langle \dot{\xi}_n, \dot{\xi}_n\rangle\,\mathrm{d}s + \int_0^1 \langle \delta(x_n), \dot{\xi}_n\rangle^2\,\mathrm{d}s = \mathrm{o}(1),$$

thus $\int_0^1 \langle \dot{\xi}_n, \dot{\xi}_n\rangle\,\mathrm{d}s = \mathrm{o}(1)$ implies $\xi_n \to 0$ strongly in $H^1(I, \mathbb{R}^N)$. □

Remark 4.2. If $P_0 \cap P_1 = \emptyset$, then by Remark 2.10 F is C^1 in all $\Gamma(P_0, P_1)$ and arguing as in Lemma 4.1 the $(\mathrm{PS})_a$ condition holds at any level $a \in \mathbb{R}$.

If $P_0 \cap P_1$ is not empty, some constants are in $\Gamma(P_0, P_1)$ and F is not differentiable at level zero then the result in Remark 3.4 is not obvious. Nevertheless the following lemma can be proved:

Lemma 4.3. *For any $c \in \mathbb{R}$ the sublevel F^c is such that*

$$\mathrm{cat}_{\Gamma(P_0,P_1)}(F^c) < +\infty. \tag{4.9}$$

Proof. If $P_0 \cap P_1 = \emptyset$, (4.9) follows by Remark 3.4. If, on the contrary, $P_0 \cap P_1 \neq \emptyset$, we consider the functional

$$g(x) = \int_0^1 \langle \dot{x}, \dot{x}\rangle\,\mathrm{d}s, \qquad x \in \Gamma(P_0, P_1).$$

It is well known that g is of class C^1; moreover, simplifying the arguments in Lemma 4.1, it can be proved that g satisfies $(\mathrm{PS})_a$ for all $a \in \mathbb{R}$. By Remark 3.4 it follows

$$\mathrm{cat}_{\Gamma(P_0,P_1)}(g^b) < +\infty \quad \text{for any } b \in \mathbb{R}. \tag{4.10}$$

As (2.12) implies that fixed $c \in \mathbb{R}$ there exists $b \in \mathbb{R}$ such that

$$F^c \subset g^b,$$

then (4.9) follows by (4.10). □

Proof of Theorem 1.1. Let $P_0 \cap P_1 = \emptyset$. By (2.12), Remark 4.2 and Theorem 3.3 it follows that

$$c = \inf_{x \in \Gamma(P_0, P_1)} F(x) > 0$$

is attained. If, moreover, P_0 and P_1 are contractible in $\mathcal{M}_0$, then by Theorems 3.3 and 3.7 the functional F has at least $\mathrm{cat}(P_0 \times P_1)$ strictly positive critical levels. Hence Theorem 2.11 can be applied. □

Proof of Theorem 1.2. Assume now that the hypotheses of Theorem 1.2 hold.

If $P_0 \cap P_1 = \emptyset$, then by (2.12), Remark 4.2 and Theorems 3.3 and 3.6 it follows that F has infinitely many strictly positive critical levels. However for finding an estimate of the "arrival times" it is better to use the following tools which work even if $P_0 \cap P_1 \neq \emptyset$.

Let $\varepsilon > 0$ be fixed. We claim there exists $\bar{k} \in \mathbb{N}$ such that

$$B \in \Gamma_{\bar{k}} \quad \Longrightarrow \quad B \cap F_\varepsilon \neq \emptyset, \tag{4.11}$$

where $\Gamma_{\bar{k}}$ is defined in (3.1) and $F_\varepsilon = \{x \in \Gamma(P_0, P_1) : F(x) > \varepsilon\}$.

In fact, if (4.11) does not hold there exists a sequence $(B_n)_{n\in\mathbb{N}}$ of subsets of $\Gamma(P_0, P_1)$ such that

$$\mathrm{cat}_{\Gamma(P_0,P_1)}(B_n) \geq n, \qquad B_n \subset F^\varepsilon \quad \text{for all } n \in \mathbb{N},$$

thus $\mathrm{cat}_{\Gamma(P_0,P_1)}(F^\varepsilon) = +\infty$ in contradiction with Lemma 4.3.

Let $\bar{k}$ be such that (4.11) holds and consider the corresponding $c_{\bar{k}}$ defined as in (3.1). By (4.11) and Theorem 3.6 it follows that

$$\Gamma_{\bar{k}} \neq \emptyset, \qquad \varepsilon \leq c_{\bar{k}} < +\infty,$$

hence by Remark 3.5 and Lemma 4.1 $c_{\bar{k}}$ is a strictly positive critical level of F.

As $\varepsilon > 0$ is fixed in an arbitrary way, it is possible to consider two sequences $\varepsilon_n \nearrow +\infty$ and $k_n \nearrow +\infty$ such that

$$0 < \varepsilon_n \leq c_{k_n} < \varepsilon_{n+1} \leq c_{k_{n+1}}$$

and for each $n \in \mathbb{N}$ there exists $x_n \in \Gamma(P_0, P_1)$ critical point of F at level c_{k_n}. By Theorem 2.11 it follows that problem (1.3) has infinitely many solutions whose arrival time is $\lambda_n = F(x_n) = c_{k_n}$ such that $\lim_{n\to+\infty} \lambda_n = +\infty$. □

Remark 4.4. In the hypotheses of Theorem 1.2 it is easy to prove that the found sequence of solutions $z_n = (x_n, t_n)$ is such that

$$\lim_{n\to+\infty} \int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle \, \mathrm{d}s = +\infty.$$

In fact, by means of simple calculations, (1.4) and (2.11) imply that there exists $\bar{c}_0 > 0$ such that

$$F(x) \leq \bar{c}_0 \left(\int_0^1 \langle \dot{x}, \dot{x} \rangle \, \mathrm{d}s \right)^{1/2} \quad \text{for all } x \in \Gamma(P_0, P_1).$$

5. Static case

Let $(\mathcal{M}, \langle \cdot, \cdot \rangle_z)$ be a static Lorentz manifold satisfying (M_0) and such that for some constants N, $\nu > 0$ there results $\nu \leq \beta(x) \leq N$ for all $x \in \mathcal{M}_0$.

Arguing as in Section 2 we can suppose that $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ is equipped with the static Lorentzian metric

$$\langle \zeta, \zeta \rangle_z = \langle \xi, \xi \rangle - \tau^2 \tag{5.1}$$

for any $z = (x, t) \in \mathcal{M}_0 \times \mathbb{R}$ and $\zeta = (\xi, \tau) \in T_z\mathcal{M} \equiv T_x\mathcal{M}_0 \times \mathbb{R}$, where $\langle \cdot, \cdot \rangle$ is the Euclidean Riemannian metric on $\mathcal{M}_0$.

The following result holds:

Corollary 5.1. *Let $(\mathcal{M}, \langle \cdot, \cdot \rangle_z)$ be a static Lorentz manifold satisfying* (M_0) *and such that for some constants N, $\nu > 0$ there results $\nu \leq \beta(x) \leq N$ for all $x \in \mathcal{M}_0$. Let P_0 and P_1 be two closed submanifolds of $\mathcal{M}_0$ such that* (C) *holds. If $P_0 \cap P_1 = \emptyset$, then there exists at least one solution of* (1.3), *while if P_0 and P_1 are both contractible in $\mathcal{M}_0$ then the solutions of* (1.3) *are at least* $\operatorname{cat}(P_0 \times P_1)$. *If either $P_0 \cap P_1 = \emptyset$ or $P_0 \cap P_1 \neq \emptyset$, and P_0 and P_1 are both contractible in $\mathcal{M}_0$ while $\mathcal{M}_0$ is not contractible in itself, then problem* (1.3) *has infinitely many non-constant solutions $z_n(s) = (x_n(s), t_n(s))$ such that the "lenght" of x_n and the "arrival time" $t_n(1)$ are diverging increasing sequences.*

Corollary 5.1 can be proved by means of Theorems 1.1 and 1.2 applied to a static Lorentz manifold, however we want to prove Corollary 5.1 by using a simpler variational approach.

Theorem 5.2. *Let $\mathcal{M}$ be a manifold endowed by the static Lorentz metric* (5.1) *and $z = (x, t)$ be a smooth curve. The following propositions are equivalent:*
(a) *$z = z(s)$ is a geodesic in $\mathcal{M}$;*
(b) *$x = x(s)$ is a geodesic in $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$ and there exists $k \in \mathbb{R}$ such that $t(s) = k\,s + t(0)$ for all $s \in I$.*
Moreover, z is a lightlike geodesic in $\mathcal{M}$ starting from $P_0 \times \{0\}$ which arrives to $P_1 \times \mathbb{R}$ if and only if (b) *holds, x joins P_0 and P_1 and*

$$t(s) = L(x)s \quad \textit{where} \quad L(x) = \int_0^1 \sqrt{\langle \dot{x}, \dot{x} \rangle} \, \mathrm{d}s \quad \textit{is the length of } x.$$

Proof. By (5.1) it is

$$\langle D_s\dot{z}(s), \zeta\rangle_z = \langle D_s\dot{x}(s), \xi\rangle - \ddot{t}(s)\tau \tag{5.2}$$

for all $s \in I$, $\zeta = (\xi, \tau) \in T_{z(s)}\mathcal{M}$. Whence (a) implies that x is a geodesic in $\mathcal{M}_0$ and there exist two constants E, $\bar{E} \in \mathbb{R}$ such that

$$E = \langle \dot{z}(s), \dot{z}(s)\rangle_z, \qquad \bar{E} = \langle \dot{x}(s), \dot{x}(s)\rangle \quad \text{for all } s \in I. \tag{5.3}$$

By (5.1) and (5.3) it follows that there exists a real constant k such that $\dot{t}(s) = k$ for any $s \in I$, hence (b) holds.

Vice versa, if x is a geodesic in $\mathcal{M}_0$ and t is a straight line, by (5.2) it follows that $D_s\dot{z} = 0$.

Now, if z is a lightlike geodesic in $\mathcal{M}$ from $P_0 \times \{0\}$ to $P_1 \times \mathbb{R}$, then x joins P_0 and P_1, $t(0) = 0$ and in (5.3) it is $E = 0$, whence $\dot{t}(s) = \sqrt{\langle \dot{x}(s), \dot{x}(s)\rangle}$ which implies $t(s) = L(x)s$. The contrary follows easily by (5.1). □

Proof of Corollary 5.1. By Theorem 5.2 it follows that searching solutions of (1.3) is equivalent to study critical points of the functional

$$G(x) = \frac{1}{2}\int_0^1 \langle \dot{x}, \dot{x}\rangle \, ds, \qquad x \in \Gamma(P_0, P_1).$$

As G is a C^1 map on $\Gamma(P_0, P_1)$ and verifies $(\mathrm{PS})_a$ condition for any $a \in \mathbb{R}$, the proof follows by Theorems 3.3, 3.6 and 3.7. □

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